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Algebraically Special Fluid Space-Times with Hypersurface-Orthogonal Shearfree Rays

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Abstract

The Einstein field equations are solved subject to the assumptions that (1) the source of the gravitational field is a non-rotating perfect fluid, (2) the Weyl tensor is algebraically special, (3) the repeated principal null direction is tangent to a geodesic, shearfree and twistfree congruence, and is parallely transferred along the fluid congruence. The solutions in which the line-element admits a multiply transitive group of motions have been studied by Stewart and Ellis; the remaining ones are new, and appear to represent inhomogeneous anisotropic cosmological models.

1. *Introduction*

Solutions of the Einstein field equations with a perfect fluid as source are of physical interest as cosmological models and as models for the interior of stars. The problem of finding exact solutions of these equations is usually approached by assuming the existence of a group G_r of motions with $r \geqslant 3$ (see for example Heckmann & Schucking, 1962; Stewart & Ellis, 1968; Ellis & MacCallum, 1969, and the many references given there). In general, the Weyl tensor of such space-times, regarded as representing the free gravitational field (Szekeres, 1966), is algebraically general. However, in the case that the group of motions is multiply transitive on some subspace, considered by Stewart & Ellis (1968), the Weyl tensor is algebraically special, in fact of type [22]. In addition the two repeated principal null directions are tangent to geodesic and shearfree null congruences (Wainwright, 1970). The solutions in which these congruences are also *twistfree* but *expanding* constitute class II of Stewart & Ellis (1968), and are of physical interest: they include the solution corresponding to the general spherically symmetric fluid source, the special class of homogeneous, anisotropic model universes studied by Kompaneets & Chernov (1964), Zel'dovich (1965), Doroshkevich (1965), Kantowski & Sachs (1966), Thorne (1967) and others [see Vajk & Eltgroth (1970) for a detailed survey], and certain inhomogeneous model universes used by Eardley, Liang & Sachs (1972) in their study of cosmological singularities.

In view of these considerations, it is natural to attempt a systematic discussion of the Einstein field equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -T_{ab} \tag{1.1}$$

subject to the following conditions:

Condition I

Space-time admits a geodesic, shearfree, twistfree, expanding null congruence with tangent field k^a, i.e.

$$k_{a;b}k^b = 0 \qquad k_{[a;b}k_{c]} = 0$$
$$(k_{a;b} + k_{b;a})k^{a;b} = (k^a{}_{;a})^2 \neq 0 \tag{1.2}$$

with respect to an affine parameter.

Condition II

The tangent field k^a in I is a repeated principal null direction of the Weyl tensor:

$$k_{[e}C_{a]bcd}k^b k^c = 0 \tag{1.3}$$

Condition III

The source of the gravitational field is a perfect fluid, with energy tensor

$$T_{ab} = (A + p)u_a u_b - pg_{ab} \tag{1.4}$$

and the fluid velocity u^a, rest energy density A and pressure p satisfying

$$u^a u_a = 1 \tag{1.5}$$

$$A + p > 0 \tag{1.6}$$

We note that the analogous sourcefree problem (i.e. $R_{ab} = 0$) was solved by Robinson & Trautman (1962): in this case condition II is a consequence of I in view of the Goldberg-Sachs theorem (see Pirani, 1964).

In order to be able to classify the solutions in the present problem, we have been compelled to impose the following two additional conditions:

Condition IV

The direction of the tangent field k^a in I is parallely transferred [see Pirani (1964), p. 325] along the fluid congruence, i.e.

$$k_{a;b}u^b = fk_a \Leftrightarrow k_{[a}k_{b];c}u^c = 0 \tag{1.7}$$

Condition V

The fluid is non-rotating, i.e.

$$u_{[a;b}u_{c]} = 0 \tag{1.8}$$

We note that both IV and V are satisfied by the subclass of solutions admitting a multiply transitive group of motions referred to earlier.

The purpose of this article is to classify the solutions of the Einstein field equations subject to conditions I-V. The procedure depends on establishing that *either the line-element admits a multiply transitive group of motions* (and hence is in class II of Stewart & Ellis, 1968) *or the expansion* $\rho \equiv \frac{1}{2}k^a{}_{;a}$ *of the preferred null congruence is given by one of the expressions*

$$\rho = -\tfrac{1}{2}r^{-1} \tag{1.9}$$

$$\rho = -r(r^2 - k^2)^{-1}, \qquad k = \text{const.} \tag{1.10}$$

where r is a suitably chosen affine parameter corresponding to the tangent field k^a *in I.* In proving this result (see Theorem 5.1) one has to make use of most of the information contained in the field equations. This is in contrast to the source-free problem (Robinson & Trautman, 1962) where it follows, using only the field equation $R_{ab}k^ak^b = 0$ (k^a is the tangent field of the preferred null congruence), that the expansion is of the form

$$\rho = -r^{-1} \tag{1.11}$$

independently of whether or not the line-element admits a multiply transitive group of motions.

The derivation of the general line element is carried out in two stages in an attempt to clarify the roles of the various conditions. In the first stage we do not make use of the field equations, but instead impose I and II and the following purely geometrical restrictions on the Ricci tensor:

*Condition III**

$$k_{[a}k_{b];d}R^d{}_c = fk_{[a}k_{b];c} \tag{1.12}$$

$$R_{ab}k^ak^b \leqslant 0 \tag{1.13}$$

where k^a is the tangent field in I, and f is some function.

This condition is compatible with the presence of a perfect fluid satisfying (1.6), but is weaker than III and IV together. In fact *if I and III hold, then (1.7) and (1.12) are equivalent,* as follows directly from (1.1) and (1.4).

The line-element characterized by conditions I, II and III*, obtained in Sections 2 and 3 using the spin coefficient formalism of Newman & Penrose (1962), will be referred to as *the generalized Robinson–Trautman line-element* (see Theorem 2.1) as it is perhaps of interest in itself. The spin coefficients, and tetrad components of the Ricci and Weyl tensors for this line-element are listed in Section 4, and in Section 5 the preceding results are used to simplify the field equations (1.1) subject to conditions I–V. In the case that the line-element admits a multiply transitive group of motions, the field equations reduce to an underdetermined system of two partial differential equations for three functions of two variables. Otherwise we obtain a system of two (respectively one) partial differential equations for two (respectively one) functions of three variables, depending on whether the expansion ρ is given by (1.9) or (1.10).

We note that one other result analogous to Theorem 5.1 is known. Oleson (1971) proved that for any solution of the Einstein fluid equations with a non-rotating perfect fluid as source, if the Weyl tensor admits a repeated principal null direction of multiplicity 4 (i.e. is of type [4]), then this direction is tangent to a geodesic and twistfree null congruence, whose expansion is given by precisely one of the expressions (1.9), (1.10); in this case however, the shear of the null congruence is non-zero.

As regards techniques for dealing with the type of problem under consideration here, the spin coefficient formalism of Newman & Penrose (1962) is particularly convenient. In the remainder of this paper familiarity with this formalism on the part of the reader is assumed. Our conventions for the Riemann, Ricci and Weyl tensors are those of Newman & Penrose.

2. *The Generalized Robinson–Trautman Line-Element*

Robinson & Trautman (1962) [see p. 465, equation (4)] have shown that for any space-time satisfying condition I the line-element can be written in the form

$$ds^2 = -\tfrac{1}{2}G^{-2}\,dz\,d\bar{z} + 2\,du(dr - \tfrac{1}{2}\bar{W}\,dz - \tfrac{1}{2}W\,d\bar{z} - U\,du) \qquad (2.1)$$

where G, U are arbitrary real functions, and W is complex. With $u = x^1$, $r = x^2$, $z = x^3 + ix^4$, the tangent field k^a of the preferred null congruence is given by

$$k^a = \delta^a_2, \qquad k_a = \delta^1_a \qquad (2.2)$$

The null geodesics thus lie in the null hypersurfaces u = constant, and r is an affine parameter along these geodesics.

In terms of this coordinate system we can write down and characterize the generalized Robinson–Trautman line-element.

Theorem 2.1. The line-element of a space-time can be written in the form

$$ds^2 = -\tfrac{1}{2}P^{-2}\chi^2\,dz\,d\bar{z} + 2\,du(dr - U\,du) \qquad (2.3)$$

with

$$U = r\partial_u \ln P + U^0(z, \bar{z}, u) + S(r, u)$$

and $\qquad$ (2.4)

$$P = P(z, \bar{z}, u), \qquad \chi = \chi(r, u)$$

subject to

$$\partial_r\chi \neq 0, \qquad \chi^{-1}\,\partial_r\,\partial_r\chi \leqslant 0 \qquad (2.5)$$

if and only if the space-time satisfies conditions I, II and III*.

The theorem states essentially that conditions I, II and III* separate out the r-dependence from the $(z, \bar{z})$-dependence, but not from the u-dependence, in the line-element. However, apart from (2.5), the specific r-dependence of χ and S is unrestricted.

For the purpose of comparison, recall that condition I together with the vacuum field equations

$$R_{ab} = 0$$

enables one to completely separate out the r-dependence. (Not all of these field equations are required. See Robinson & Trautman, 1962; Trim & Wainwright, 1971). In fact under these conditions, the r-dependence of the line-element is explicitly determined according to

$$\chi = r, \qquad S = m(u)r^{-1}$$

(see Robinson & Trautman, 1962).

3. *Proof of the Theorem*

The proof of Theorem 2.1 is based on the spin coefficient formalism of Newman & Penrose (1962), with which we assume the reader is familiar. The Ricci and Bianchi identities in their most general form, as given in Pirani (1964), are labelled I_1–I_{18} and II_1–II_9 respectively. For convenience we list the commutators:

$$\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta \qquad (3.1)$$

$$\delta D - D\delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \qquad (3.2)$$

$$\delta\Delta - \Delta\delta = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta \qquad (3.3)$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta \qquad (3.4)$$

Our starting point is the line-element (2.1) which, as mentioned in Section 2, is a consequence of I. A convenient null tetrad for this line element is obtained by choosing

$$\begin{aligned} n^a &= \delta_1^a + U\delta_2^a, \\ m^a &= G(\delta_3^a + i\delta_4^a + W\delta_2^a) \end{aligned} \qquad (3.5)$$

together with the preferred null vector field k^a as given by (2.2). Firstly, condition I restricts the spin coefficients associated with this tetrad according to

$$\kappa = \sigma = \rho - \bar{\rho} = \epsilon + \bar{\epsilon} = 0, \qquad \rho \neq 0 \qquad (3.6)$$

Secondly, condition II restricts the Weyl tensor tetrad components according to

$$\Psi_0 = \Psi_1 = 0 \qquad (3.7)$$

Finally a straightforward calculation shows that when I is satisfied condition III* is equivalent to

$$\bar{\tau}\Phi_{00} - \rho\Phi_{10} = 0, \qquad \bar{\tau}\Phi_{10} - \rho\Phi_{20} = 0 \qquad (3.8)$$

$$\bar{\tau}^2\Phi_{01} - 2\rho\tau\Phi_{11} + \rho^2\Phi_{21} = 0 \qquad (3.9)$$

$$\Phi_{00} \geqslant 0 \qquad (3.10)$$

The specific form (2.2), (3.5) of the null tetrad places restrictions on the spin coefficients in addition to (3.6). In fact, applying the commutators (3.1)-(3.4) to $\phi = x^1, x^2, x^3, x^4$ successively, one finds

$$\epsilon = \lambda = \tau + \bar{\pi} = \bar{\tau} - \alpha - \bar{\beta} = 0 \tag{3.11}$$

$$\mu - \bar{\mu} = 2(\gamma - \bar{\gamma}) \tag{3.12}$$

together with the following equations relating the non-zero spin coefficients to the metric components:

$$GDW = -2\tau, \qquad G(\bar{\delta}W - \delta\bar{W}) = -(\mu - \bar{\mu}) \tag{3.13}$$

$$DG = \rho G, \qquad \Delta G = -\tfrac{1}{2}(\mu + \bar{\mu})G, \qquad \bar{\delta}G = (\alpha - \bar{\beta})G \tag{3.14}$$

$$DU = -(\gamma + \bar{\gamma}), \qquad \delta U - G\Delta W = -\bar{\nu} \tag{3.15}$$

To proceed further we need the general coordinate transformation which preserves the form (2.1) of the line-element. This has been given by Robinson & Trautman (1962) (see p. 465) as

$$\begin{aligned} u' &= f(u) \\ r' &= [f'(u)]^{-1} r + g(z, \bar{z}, u) \\ z' &= h(z, u) \end{aligned} \tag{3.16}$$

In order to preserve the form (2.2), (3.5) of the null tetrad, this must be combined with a tetrad transformation of the form

$$\begin{aligned} k^{a*} &= Rk^a \\ n^{a*} &= R^{-1}n^a - Tm^a - \overline{Tm}^a + RT\bar{T}k^a \\ m^{a*} &= e^{iS}(m^a - R\bar{T}k^a) \end{aligned} \tag{3.17}$$

with

$$R = f'(u), \qquad e^{iS} = \left(\frac{\partial h/\partial z}{\partial \bar{h}/\partial \bar{z}}\right)^{1/2} \tag{3.18}$$

and with T related to $h(z, u)$ according to

$$T\bar{\delta}h + \Delta h = 0 \tag{3.19}$$

as is easily verified. Note that by (3.16), h has to satisfy

$$Dh = \delta h = 0 \tag{3.20}$$

It is easily verified using (3.5), (3.11) and (3.14) that the system of equations (3.19), (3.20) has a non-trivial solution for h provided that the function T satisfies

$$DT + \rho T = 0, \qquad \delta T + (2\bar{\alpha} - \tau)T = 0 \tag{3.21}$$

The freedom contained in (3.16)-(3.19) can in fact be used to transform the function W in the line-element to zero. Under (3.16) with $f'(u) = 1$, W transforms according to

$$W' = W + \partial_{\bar{z}} g \tag{3.22}$$

where

$$2\partial_z = \partial/\partial x^3 - i\partial/\partial x^4, \qquad 2\partial_{\bar{z}} = \partial/\partial x^3 + i\partial/\partial x^4$$

Since $g = g(u, z, \bar{z})$ the integrability conditions for $W' = 0$ are

$$\partial_r W = 0, \qquad \partial_z W - \partial_{\bar{z}} \overline{W} = 0$$

i.e.

$$DW = 0, \qquad \bar{\delta} W - \delta \overline{W} = 0 \tag{3.23}$$

in terms of the Newman-Penrose differential operators. Thus, by (3.13), we have to show that τ and $\mu - \bar{\mu}$ can be transformed to zero.

Under (3.17) τ transforms according to

$$\tau^* = e^{iS}(\tau + T\rho)$$

Using the Ricci identities I_1, I_3, I_{11} and I_{16}, and the assumptions (3.7) and (3.8), it follows that the choice $T = -\tau/\rho$ satisfies the conditions (3.21). Thus we may use the combined coordinate and tetrad freedom to transform

$$\tau = 0 \Rightarrow \alpha + \bar{\beta} = 0 \tag{3.24}$$

This is one of the steps of the proof where assumption III* plays a crucial role. The Ricci identities I_3 and I_{16}, with (3.7), now imply

$$\Phi_{01} = \Phi_{02} = 0 \tag{3.25}$$

Furthermore, by I_{11}

$$\delta\rho = 0 \tag{3.26}$$

which when substituted in the commutator (3.4) (using $\rho = \bar{\rho}$) implies

$$(\mu - \bar{\mu})D\rho = 0$$

However from I_1, using (3.10) and the fact that $\rho \neq 0$, one concludes that $D\rho \neq 0$, so that

$$\mu = \bar{\mu} \Rightarrow \gamma = \bar{\gamma} \tag{3.27}$$

Equations (3.13) thus reduce to the integrability conditions (3.23), and we can use the coordinate freedom to transform

$$W = 0 \tag{3.28}$$

Note that in order to preserve (3.24) and (3.28), the combined tetrad and coordinate freedom (3.16), (3.17) must be restricted by

$$\partial_u h = 0, \qquad \partial_z g = \partial_{\bar{z}} g = 0 \Rightarrow T = 0 \tag{3.29}$$

The key to the remainder of the proof (and in fact the remainder of the paper) is the introduction of a *real* potential function

$$\chi = \chi(r, u)$$

such that the expansion is given by

$$\rho = -\chi^{-1} \partial_r \chi \tag{3.30}$$

This is made possible by the condition (3.26), which with (3.28) ensures that $\rho = \rho(r, u)$. Note that χ is not uniquely determined: the freedom in the choice of χ is described by the transformation

$$\hat{\chi} = l(u) . \chi \tag{3.31}$$

The first of equations (3.14) can now be integrated to yield

$$G = P\chi^{-1} \tag{3.32}$$

where P is an arbitrary function of u, z, $\bar{z}$. In addition, using (3.30) the Ricci identity I_1 implies

$$\Phi_{00} = -\chi^{-1} \partial_r \partial_r \chi \tag{3.33}$$

so that χ satisfies the second of conditions (2.5) by virtue of the assumption (3.10).

In order to complete the proof of the sufficiency, we have to show that the function U has the form (2.4). This is essentially a consequence of the condition (3.9), which, with (3.24), simplifies to

$$\Phi_{12} = 0 \tag{3.34}$$

Thus, using equations (3.11), (3.24) and (3.27), the Ricci identity I_{15} reduces to

$$\delta\gamma + \Delta\bar{\alpha} = -\mu\bar{\alpha} \tag{3.35}$$

Furthermore, by means of (3.14), (3.15), (3.24) and (3.27) the spin coefficients α, γ and μ can be written in the form

$$\begin{aligned} \alpha = \chi^{-1} \partial_{\bar{z}} P, \qquad \gamma = -\tfrac{1}{2} \partial_r U \\ \mu = -\rho U - \partial_u \ln P + \partial_u \ln \chi \end{aligned} \tag{3.36}$$

Using equations (3.32) and (3.36), equation (3.35) reduces to

$$\partial_z \partial_r (U - r \partial_u \ln P) = 0$$

Since $U - r \partial_u \ln P$ is real, this implies that U is of the form (2.4), as required. Note that the decomposition (2.4) for U does not define U^0 and S uniquely. The freedom in chioce of U^0 and S is described by the transformation

$$\begin{aligned} \hat{S} = S + s(u) \\ \hat{U}^0 = U^0 - s(u) \end{aligned} \tag{3.37}$$

This completes the proof of the sufficiency.

The proof of the necessity is essentially trivial. Assume a line element of the form (2.3)–(2.5). A suitable null tetrad is

$$k^a = \delta^a_2, \qquad n^a = \delta^a_1 + U\delta^a_2$$
$$m^a = P\chi^{-1}(\delta^a_3 + i\delta^a_4)$$

One can express the spin coefficients in terms of P, χ and U using the commutators, and the tetrad components of the Weyl and Ricci tensors in terms of the spin coefficients using the appropriate Ricci identities. One finds that conditions I, II, III* hold, in the form (3.6)–(3.10).

4. *The Weyl and Ricci Tensors*

The Ricci identities which are not satisfied identically by the line-element (2.3)–(2.5), namely $I_1, I_6, I_9, I_{10}, I_{12}, I_{14}, I_{17}$ express the non-zero components of the Weyl and Ricci tensors, namely $\Psi_2, \Psi_3, \Psi_4, \Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda$ in terms of the spin coefficients, which in turn are expressed in terms of the metric components. Before listing these formulae, we summarize the result obtained so far for the generalized Robinson–Trautman line-element of Section 2. The null tetrad (2.2), (3.5) has been simplified to the form

$$k^a = \delta^a_2, \qquad n^a = \delta^a_1 + U\delta^a_2$$
$$m^a = P\chi^{-1}(\delta^a_3 + i\delta^a_4) \tag{4.1}$$

with U given by (2.4). The spin coefficients associated with this tetrad are

$$\kappa = \sigma = \rho - \bar{\rho} = 0 \tag{4.2}$$

$$\epsilon = \pi = \tau = \lambda = \alpha + \bar{\beta} = \mu - \bar{\mu} = \gamma - \bar{\gamma} = 0 \tag{4.3}$$

$$\rho = -\partial_r \ln \chi, \qquad \alpha = \chi^{-1}\, \partial_z P \tag{4.4}$$

$$\gamma = -\tfrac{1}{2}(\partial_u \ln P + \partial_r S) \tag{4.5}$$

$$\mu = (-1 + r\, \partial_r \ln \chi)\partial_u \ln P + \partial_r \ln \chi (U^0 + S) + \partial_u \ln \chi \tag{4.6}$$

$$\nu = -2P\chi^{-1}(r\, \partial_z\, \partial_u \ln P + \partial_z U^0) \tag{4.7}$$

These expressions are obtained from (3.30), (3.36) and (3.15), using (3.28), (3.32) and (2.4).

The Ricci tensor components are

$$\Phi_{01} = \Phi_{02} = \Phi_{12} = 0 \tag{4.8}$$

$$\Phi_{00} = -\chi^{-1}\, \partial_r\, \partial_r\, \chi \tag{4.9}$$

$$\begin{aligned} 2\Phi_{11} = &-\tfrac{1}{2}\, \partial_r\, \partial_r S + \rho(\rho r + 1)\, \partial_u \ln P + \rho^2(S + U^0) \\ &-\rho\, \partial_u \ln \chi + \chi^{-2} K \end{aligned} \tag{4.10}$$

$$\begin{aligned} 6\Lambda = &\tfrac{1}{2}\, \partial_r\, \partial_r S - 2\rho\, \partial_r S + (U^0 + S)(\rho^2 - 2\Phi_{00}) + \chi^{-2} K \\ &+ [\rho + r(\rho^2 - 2\Phi_{00})]\, \partial_u \ln P + \chi^{-2}(2\chi\, \partial_u\, \partial_r \chi + \partial_u \chi\, \partial_r \chi) \end{aligned} \tag{4.11}$$

$$\Phi_{22} = U^2\Phi_{00} + P^2[-r\chi^{-2}\,\partial_u(P^{-2}K) + \rho\,\partial_u(P^{-2}U^0)$$
$$-4\chi^{-2}\,\partial_z\,\partial_{\bar z}U^0 - \tfrac{1}{2}(1+r\rho)\,\partial_u\,\partial_u P^{-2}] - 2U^0\chi^{-2}\,\partial_u\,\partial_r\chi$$
$$+\partial_u \ln P(-\partial_r S - 2\rho S + 3\,\partial_u \ln\chi - 2r\chi^{-1}\,\partial_r\,\partial_u\chi)$$
$$+\chi^{-1}(-\partial_r\chi\,\partial_u S - 2S\,\partial_u\,\partial_r\chi - \partial_u\,\partial_u\chi + \partial_u\chi\,\partial_r S) \qquad (4.12)$$

with

$$K = 4P^2\,\partial_z\,\partial_{\bar z}\ln P \qquad (4.13)$$

The Weyl tensor components are

$$\Psi_0 = \Psi_1 = 0 \qquad (4.14)$$
$$\Psi_2 = -\tfrac{1}{2}\,\partial_r\,\partial_r S + \Lambda - \Phi_{11} \qquad (4.15)$$
$$\Psi_3 = -2\chi^{-1}P[\rho\,\partial_z U^0 + (1+r\rho)\,\partial_z\,\partial_u \ln P] \qquad (4.16)$$
$$\Psi_4 = -4\chi^{-2}[r\,\partial_z(P^2\,\partial_z\,\partial_u \ln P) + \partial_z(P^2\,\partial_z U^0)] \qquad (4.17)$$

Note that the dependence of Ψ_3 and Ψ_4 on the affine parameter r along the preferred null congruence is completely determined by the potential function $\chi(r, u)$, defined by equation (3.30).

To conclude this section we summarize the combined coordinate and tetrad freedom which preserves the form of the line-element (2.3)-(2.5) and the tetrad (4.1):

$$u' = f(u) \qquad (4.18a)$$
$$r' = [f'(u)]^{-1}r + g(u) \qquad (4.18b)$$
$$z' = h(z) \qquad (4.18c)$$
$$k^{a*} = f'(u)\,k^a \qquad (4.19a)$$
$$n^{a*} = [f'(u)]^{-1}n^a \qquad (4.19b)$$
$$m^{a*} = \left[\frac{h'(z)}{\bar h'(\bar z)}\right]^{1/2} m^a \qquad (4.19c)$$

These equations follow from (3.16)-(3.18) and (3.29).

5. *Perfect Fluid Space-Times*

We are now in a position to return to the problem posed in the introduction: *solve the Einstein field equations (1.1) subject to the conditions I-V.* The following theorem enables one to deal systematically with this problem.

Theorem 5.1. For any solution of the Einstein field equations satisfying conditions I-V, the expansion of the preferred null congruence is given, in terms of a suitably chosen affine parameter r along the congruence, by

$$\rho = -\tfrac{1}{2}r^{-1}$$

or

$$\rho = -r(r^2 - k^2)^{-1}, \qquad k = \text{const.}$$

provided that the line-element does not admit a multiply transitive group of motions.

Proof. As mentioned in Section 1, conditions I–V imply conditions I, II and III*. We can thus use the results of Sections 2 to 4, by requiring that the Ricci tensor of the generalized Robinson–Trautman line-element (2.3)-(2.5), as given by equations (4.8)-(4.12), be algebraically compatible with the perfect fluid energy tensor (1.4) in the field equations (1.1).

Firstly, since $\Phi_{01} \equiv -\frac{1}{2}R_{ab}k^a m^b = 0$ [by (4.8)], we conclude from (1.1) and (1.4) that the fluid velocity u^a satisfies

$$u^a m_a = 0$$

On account of (1.5), u^a can thus be expressed in terms of the null tetrad as

$$u^a = 2^{-1/2}(Bk^a + B^{-1}n^a) \tag{5.1}$$

for some positive function B. (The vectors k^a and n^a are chosen to be future-pointing null; the requirement that B is positive ensures that the timelike vector u^a is also future-pointing). The conditions $\Phi_{02} \equiv -\frac{1}{2}R_{ab}m^a m^b = 0$ and $\Phi_{12} \equiv -\frac{1}{2}R_{ab}m^a n^b = 0$, are then satisfied identically. In addition equation (5.1) [with (1.1) and (1.4)] implies that the non-zero components Φ_{00}, Φ_{11} and Φ_{22} must be related according to

$$B^2\Phi_{00} = 2\Phi_{11} = B^{-2}\Phi_{22} \tag{5.2}$$

Once these conditions are algebraically satisfied, the density A and pressure p can be calculated using

$$A + p = 8\Phi_{11}, \qquad A - 3p = 24\Lambda \tag{5.3}$$

as follows from (1.1) and (1.4).

Condition V (which has as yet not been used) imposes a strong restriction on the form of the function B^2. Using (5.1), (4.1) and (2.2) one finds that

$$u_a = 2^{-1/2}B^{-1}[(B^2 - U)\delta_a^1 + \delta_a^2] \tag{5.4}$$

Using (1.8), it follows that condition V is equivalent to

$$\partial_z(B^2 - U) = 0 = \partial_{\bar z}(B^2 - u)$$

One concludes that

$$B^2 = U + F(r, U) \tag{5.5}$$

i.e. *the* $(z, \bar z)$*-dependence of* B^2 *is the same as that of* U. Equation (5.4) assumes the form

$$u_a = 2^{-1/2}B^{-1}(F\delta_a^1 + \delta_a^2) \tag{5.6}$$

The first of the compatibility conditions (5.2) can now be written as

$$(U + F)\Phi_{00} = 2\Phi_{11}$$

Using the expressions (4.9) and (4.10) for Φ_{00} and Φ_{11}, one obtains, on rearranging

$$0 = \tfrac{1}{2}U^0\,\partial_r\,\partial_r\chi^2 + \tfrac{1}{2}\,\partial_u\,\ln P[-\partial_r\chi^2 + r\,\partial_r\,\partial_r\chi^2] + K$$
$$+ [-\tfrac{1}{2}\chi^2\,\partial_r\,\partial_r S + \tfrac{1}{2}S\,\partial_r\,\partial_r\chi^2 + F\chi\,\partial_r\,\partial_r\chi + \partial_r\chi\,.\,\partial_u\chi] \qquad (5.7)$$

On differentiating this equation successively with respect to z and r, only the first two terms survive. One finds the key restriction

$$(r\,\partial_z\,\partial_u\,\ln P + \partial_z U^0)\,.\,\partial_r\,\partial_r\,\partial_r\chi^2 = 0 \qquad (5.8)$$

We now show that *the requirement that space-time does not admit a multiply transitive group of motions implies that*

$$|\partial_z\,\partial_u\,\ln P| + |\partial_z U^0| \neq 0 \qquad (5.9)$$

Suppose on the contrary that

$$\partial_z\,\partial_u\,\ln P = 0 = \partial_z U^0 \qquad (5.10)$$

Since P is real, it must be of the form $p(u)\,.\,q(z, \bar{z})$. On account of the freedom (3.31) in the choice of χ, the function $p(u)$ may be absorbed in χ in the equation (3.32) defining P, and we obtain

$$\partial_u P = 0 \qquad (5.11)$$

It follows from (5.7), (5.8) and (5.11) that the quantity K [as defined by (4.13)] is constant. This means that the 2-spaces r = const., u = const. are spaces of constant curvature and we conclude (Eisenhart, 1961) that space-time admits a group of motions multiply transitive on these subspaces.

With (5.9), equation (5.8) asserts that χ^2 is quadratic in r:

$$\chi^2 = ar^2 + 2br + c \qquad (5.12)$$

where a, b, c are functions of u. Using (4.4) and (4.9) one immediately finds that

$$\rho = -(ar + b)\chi^{-2}, \qquad \Phi_{00} = (b^2 - ac)\chi^{-4} \qquad (5.13)$$

Thus in the presence of a perfect fluid satisfying (1.6) [which is equivalent to $\Phi_{00} > 0$ on account of (5.2) and (5.3)] we have

$$\Sigma \equiv b^2 - ac > 0 \qquad (5.14)$$

Two cases have to be distinguished.

Case 1: $a = 0 \Rightarrow b \neq 0$

The coordinate freedom represented by $g(u)$ in (4.18b), and the freedom (3.31) in choice of χ, can be used to set

$$c = 0, \qquad b = \tfrac{1}{2}\epsilon \qquad (5.15)$$

where

$$\begin{aligned} \epsilon = 1 &\Leftrightarrow r > 0 \\ \epsilon = -1 &\Leftrightarrow r < 0 \end{aligned} \tag{5.16}$$

In this case, by (5.12) and (5.13),

$$\chi^2 = \epsilon r, \qquad \rho = -\tfrac{1}{2} r^{-1} \tag{5.17}$$

The remaining coordinate and tetrad freedom is given by (4.18), (4.19) with $g(u) = 0$.

Case 2: $a \neq 0$

On account of (5.14) the quadratic (5.12) has two distinct real roots. Using the coordinate freedom represented by $f(u)$ and $g(u)$ in (4.18) and the freedom (3.31) in choice of χ, one can achieve

$$a = \epsilon, \qquad b = 0, \qquad c = -\epsilon k^2 \tag{5.18}$$

with $k \neq 0$ a constant and

$$\begin{aligned} \epsilon = 1 &\Leftrightarrow r^2 > k^2 \\ \epsilon = -1 &\Leftrightarrow r^2 < k^2 \end{aligned} \tag{5.19}$$

In this case

$$\chi^2 = \epsilon (r^2 - k^2), \qquad \rho = -r(r^2 - k^2)^{-1} \tag{5.20}$$

The remaining coordinate and tetrad freedom in this case is given by (4.18), (4.19) with $g(u) = 0$, $f''(u) = 0$. This concludes the proof of Theorem 5.2.

We now complete the simplification of the compatibility conditions (5.2), subject to (5.9). Until the final stage, the procedure can be carried out independently of the separation into cases 1 and 2. The equations are considerably simplified by the fact that we can obtain

$$\partial_u \chi = 0 \tag{5.21}$$

[see (5.17) and (5.20).] Until further notice, we will use χ^2 in the form (5.12), with *a, b, c* taken to be constants. [The specific values (5.15) and (5.18) will be used later.]

The first step is to substitute (5.12) into (5.7). On taking (5.9) and (5.21) into account, we conclude that

$$aU^0 - b\,\partial_u \ln P + K = G(u) \tag{5.22}$$

and

$$\tfrac{1}{2}\,\partial_r(\chi^2\,\partial_r S - S\,\partial_r \chi^2) + F\Sigma\chi^{-2} = G(u) \tag{5.23}$$

where $G(u)$ is an arbitrary function of *u*.

Equation (5.23) determines the r-dependence of $S(r, u)$ in terms of that of $F(r, u)$. The latter is determined by the second and final compatibility condition contained in (5.2), which with (5.5) reads

$$\Phi_{22} = (U+F)^2\Phi_{00}$$

Using equations (2.4), (4.9), (4.12), (5.12), (5.21) and (5.22) this can be written in the form:

$$0 = \partial_u \ln P(-\chi^2\, \partial_r S + S\, \partial_r \chi^2 + 2rG) - 2(r\, \partial_u \ln P + U^0)F\Sigma\chi^{-2}$$
$$- \Omega + [-\Sigma(2S+F)F\chi^{-2} - r\, \partial_u G - (ar+b)\partial_u S] \tag{5.24}$$

where

$$\Omega = P^2[4\, \partial_z\, \partial_{\bar{z}} U^0 + b\, \partial_u (P^{-2}U^0) + \tfrac{1}{2}c\, \partial_u\, \partial_u P^{-2}] \tag{5.25}$$

Apply ∂_z and ∂_r successively to (5.24). Only the first two terms survive. On making use of (5.22), the remaining terms can be rearranged to yield

$$\Sigma(r\, \partial_z\, \partial_u \ln P + \partial_z U^0)\, \partial_r (F\chi^{-2}) = 0$$

On account of (5.9) and (5.14) this implies

$$F(r, u) = H(u)\chi^2 \tag{5.26}$$

We can now immediately write down a first integral of the differential equation (5.23):

$$\chi^2\, \partial_r S - S\, \partial_r \chi^2 = 2(G - H\Sigma)r + m(u) \tag{5.27}$$

where m is an arbitrary function of u. On rearranging the left-hand side, we see that S can be written in the form

$$S = \chi^2 S^0(u) + 2(G - H\Sigma)\chi^2 \int r\chi^{-4}\, dr + m\chi^2 \int \chi^{-4}\, dr \tag{5.28}$$

Substitute (5.26) and (5.27) into (5.24). It follows that

$$\Omega + 2U^0 H\Sigma + m\, \partial_u \ln P + M(u) = 0 \tag{5.29}$$

where

$$M(u) = (ar+b)\, \partial_u S + H\Sigma(2S + H\chi^2) + r\, \partial_u G \tag{5.30}$$

Equation (5.30) must be satisfied identically in r: this imposes restrictions on the functions m, G, H and S^0 which appear in the expression (5.28) for S. The exact form of these restrictions depends on whether one is in case 1 or 2. In both cases these functions are either constant, or can be made to vanish using the remaining coordinate freedom (where applicable) and freedom (3.37) in choice of S and U^0. In case 1 we obtain

$$H = m = 0, \qquad \partial_u G = \partial_u S^0 = 0 \Rightarrow M(u) = 0 \tag{5.31}$$

and in case 2,

$$\left.\begin{array}{lll} G = 0, & Hm = 0, & H(H + 2S^0) = 0 \\ \partial_u m = \partial_u H = \partial_u S^0 = 0 & & \end{array}\right\} \Rightarrow M(u) = 2\epsilon H^2\Sigma^2 \tag{5.32}$$

The details are omitted.

In summary, the functions χ and S are in both cases completely determined (in terms of constants) as functions of r alone. The functions P and U^0 depend in general on z, $\bar{z}$ and u, and are restricted by the partial differential equations (5.22) and (5.29) [with (5.25)]. The results are stated in the following section.

6. *The Line-Elements and Reduced Field Equations*

In this section we give the line-element and reduced field equations (i.e. remaining unsolved field equations) for cases 1 and 2 of Section 5. *These cases contain the general solution of the Einstein field equations subject to conditions I–V, and the requirement that the metric does not admit a multiply transitive group of motions.* As remarked in Section 1 the solutions for which the line-element does admit a multiply transitive group of motions [characterized by conditions (5.10)] comprise class II of Stewart & Ellis (1968). The general line-element and reduced field equations for these solutions could easily be obtained from the results of Sections 2, 4 and 5. We omit the details as these results have been given by the above authors, though in a coordinate system which differs from the one used here.

Case 1: $\rho = -\frac{1}{2}r^{-1}$

The line-element is given [see (2.3), (2.4), (5.15), (5.17), (5.28)] by

$$ds^2 = -\tfrac{1}{2}P^{-2}\epsilon r\, dz\, d\bar{z} + 2\, du(dr - U\, du) \tag{6.1}$$

with

$$U = r\, \partial_u \ln P + U^0 + S$$

$$S = \epsilon r S^0 + 2\epsilon G r \ln |r|$$

and S^0, G being constants. The quantity ϵ equals +1 or −1 depending on whether r is in the range $r > 0$ or $r < 0$ [see (5.16)]. The function $P(z, \bar{z}, u)$ is to satisfy

$$8P^2\, \partial_z\, \partial_{\bar{z}} \ln P - \epsilon\, \partial_u \ln P = 2G \tag{6.2}$$

[see (5.22) and (4.13)]. When P is determined, the function $U^0(z, \bar{z}, u)$ is obtained as a solution of

$$8\, \partial_z\, \partial_{\bar{z}} U^0 + \epsilon\, \partial_u(U^0 P^{-2}) = 0 \tag{6.3}$$

[see (5.29), (5.31) and (5.25)].

The fluid velocity is given by

$$u_a = (2U)^{-1/2}\delta_a^2 \tag{6.4}$$

[see (5.6), (5.26) and (5.31)], and the energy density and pressure by

$$\begin{aligned} p &= r^{-1}(\tfrac{1}{2}\, \partial_u \ln P - 4\epsilon G - \tfrac{1}{2}\epsilon S^0) + \tfrac{1}{2}r^{-2}U^0 - \epsilon G r^{-1} \ln |r| \\ A - p &= 2\epsilon r^{-1}(4G + S^0) + 4\epsilon G r^{-1} \ln |r| \end{aligned} \tag{6.5}$$

[see (5.3), (4.10) and (4.11)]. Finally the non-zero Weyl tensor components, as given by (4.15)-(4.17), assume the form

$$\begin{aligned}\Psi_2 &= -\tfrac{1}{3}\epsilon(G + \tfrac{1}{2}\epsilon\,\partial_u \ln P)r^{-1} - \tfrac{1}{6}U^0 r^{-2}\\ \Psi_3 &= (\epsilon r)^{-1/2}P(-\partial_z\,\partial_u \ln P + r^{-1}\,\partial_z\,U^0)\\ \Psi_4 &= -4\epsilon[\partial_z(P^2\,\partial_z\,\partial_u \ln P) + r^{-1}\,\partial_z(P^2\,\partial_z\,U^0)]\end{aligned} \tag{6.6}$$

Case 2: $\rho = -r(r^2 - k^2)^{-1}$

The line-element is given [according to (2.3), (2.4), (5.18), (5.19), (5.22), (5.20), (5.28) and (5.32)] by

$$ds^2 = -\tfrac{1}{2}P^{-2}\chi^2\,dz\,d\bar{z} + 2\,du(dr - U\,du) \tag{6.7}$$

with

$$\begin{aligned}\chi^2 &= \epsilon(r^2 - k^2)\\ U &= r\,\partial_u \ln P - \epsilon K + S\\ S &= \chi^2\left(S^0 + m\int \chi^{-4}\,dr\right) + \epsilon k^2 H\\ K &= 4P^2\,\partial_z\,\partial_{\bar{z}} \ln P\end{aligned}$$

and S^0, m, k, H are constants satisfying the algebraic constraints

$$Hm = 0, \qquad H(H + 2S^0) = 0$$

The function $P(z, \bar{z}, u)$ is to satisfy the partial differential equation

$$P^2(4\,\partial_z\,\partial_{\bar{z}}K + \tfrac{1}{2}k^2\,\partial_u\,\partial_u\,P^{-2}) - \epsilon m\,\partial_u \ln P + 2Hk^2(K - k^2 H) = 0 \tag{6.8}$$

[see (5.22), (5.25), (5.29) and (5.32)].

The fluid velocity is given [see (5.6) and (5.26)] by

$$u_a = (2B^2)^{-1/2}(\chi^2 H\delta_a^1 + \delta_a^2) \tag{6.9}$$

with

$$B^2 = U + \chi^2 H$$

and the fluid pressure and energy density [see (5.3), (4.10) and (4.11)] by

$$\begin{aligned}p &= 2k^2\chi^{-4}(r\,\partial_u \ln P - \epsilon K + \epsilon k^2 H) - 2S^0\chi^{-2}(3r^2 - 2k^2)\\ &\quad -2m\chi^{-2}\left\{\epsilon r\chi^{-2} + (3r^2 - 2k^2)\int \chi^{-4}\,dr\right\}\\ A - p &= 4S^0\chi^{-2}(3r^2 - k^2) + 4k^2 H\chi^{-2}\\ &\quad + 4m\chi^{-2}\left\{\epsilon r\chi^{-2} + (3r^2 - k^2)\int \chi^{-4}\,dr\right\}\end{aligned} \tag{6.10}$$

Finally equations (4.15)-(4.17) for the non-zero Weyl tensor components yield

$$\begin{aligned}\Psi_2 &= \tfrac{1}{3}\chi^{-4}[r(\epsilon m - 2k^2\,\partial_u \ln P) + 2\epsilon k^2 K] - \tfrac{1}{3}\chi^{-4}(r^2 + k^2)\epsilon k^2 H\\ \Psi_3 &= 2\epsilon P\chi^{-3}[-r\epsilon\,\partial_z K + k^2\,\partial_z\,\partial_u \ln P]\\ \Psi_4 &= -4\chi^{-2}[r\,\partial_z(P^2\,\partial_z\,\partial_u \ln P) - \epsilon\,\partial_z(P^2\,\partial_z K)]\end{aligned} \tag{6.11}$$

Note that if one sets

$$k = 0 = S^0 \Rightarrow \epsilon = 1$$

the line-element in case 2 reduces to the Robinson–Trautman (1962) *vacuum line-element, and the partial differential equation* (6.8) *yields the corresponding reduced vacuum field equation.*

In both cases $\Psi_2 = 0$ implies $\Psi_3 = \Psi_4 = 0$ [see (6.6) and (6.10)]. This means that the Weyl tensor is either of type [211] or [22], or is zero. There are no dust solutions, since $p = 0$ implies $A = 0$.

7. *Discussion*

Some insight into a possible interpretation of the solutions of Section 6 is obtained by considering those solutions of case 1 for which

$$G = 0, \qquad \partial_u \ln P = 0, \qquad \epsilon = 1 \tag{7.1}$$

The pressure and energy density are then given by

$$p = -\tfrac{1}{2} S^0 r^{-1} + \tfrac{1}{2} U^0 r^{-2}$$
$$A - p = 2S^0 r^{-1}$$

In order to ensure $A - p > 0$ we choose $S^0 > 0$, and in order that p be positive for at least some $r(> 0)$, we require $U^0 > 0$.

By means of a constant change of scale along the preferred null congruence [see (4.18b) with $f''(u) = g(u) = 0$] we set

$$S^0 = 2$$

and write

$$U^0 = 2b^2, \qquad b > 0$$

By virtue of (7.1), equation (6.2) implies $\partial_z \partial_{\bar{z}} \ln P = 0$. We can thus use the coordinate freedom (4.18c) to achieve $P = 2^{-1/2}$. The line element (6.1) then assumes (with $z = x + iy$) the form

$$ds^2 = -r(dx^2 + dy^2) + 2\,du\,(dr - U\,du) \tag{7.2}$$

with

$$U = 2(b^2 + r), \qquad r > 0$$
$$u_a = (2U)^{-1/2} \delta_a^2$$
$$p = -r^{-1} + b^2 r^{-2}, \qquad A - p = 4r^{-1} \tag{7.3}$$

Note that the fluid congruence is orthogonal to the hypersurfaces $r = \text{const}$. Since $2\,\partial_z = \partial_x - i\,\partial_y$, the differential equation for U^0 assumes the form

$$(\partial_x \partial_x + \partial_y \partial_y + \partial_u) b^2 = 0 \tag{7.4}$$

[Non-trivial (positive) solutions certainly exist, for example,

$$b^2 = \alpha^2 \cosh \beta x \,.\, \cosh \gamma y \,.\, e^{-(\beta^2+\gamma^2)u} \tag{7.5}$$

with α, β, γ constants.]

The Weyl tensor components (6.6) simplify to

$$\Psi_2 = -\tfrac{1}{3}b^2 r^{-2}, \qquad \Psi_3 = 2^{1/2}\,\partial_z b^2 r^{-3/2}, \qquad \Psi_4 = -4\,\partial_z\,\partial_z b^2 r^{-1} \tag{7.6}$$

In addition we note that the acceleration, shear and expansion of the fluid are given (see Stewart & Ellis, 1968, p. 1072) by

$$\begin{aligned} \dot{u}^a \dot{u}_a &= -\tfrac{1}{32}(b^2+r)^{-2}[(b^2+r)^{-1}(\partial_u b^2)^2 + 16r^{-1}\,\partial_z b^2\,\partial_{\bar{z}} b^2] \\ \sigma_{ab}\sigma^{ab} &= \tfrac{1}{24}(b^2+r)^{-1}[\{4r^{-1}b^2 + (b^2+r)^{-1}\,\partial_u b^2\}^2 \\ &\quad + 3r^{-1}(b^2+r)^{-1}\,\partial_z b^2\,\partial_{\bar{z}} b^2] \\ \theta &= \tfrac{1}{2}(b^2+r)^{-1/2}[6 + 4r^{-1}b^2 - \tfrac{1}{2}(b^2+r)^{-1}\,\partial_u b^2] \end{aligned} \tag{7.7}$$

The simplest solution arises when

$$b^2 = \text{const.}$$

The line-element then admits a multiply transitive group of motions [see (5.10]. In this case the coordinate transformation

$$\begin{aligned} u &= \tfrac{1}{2}Z - \ln(T+b) \\ r &= T(T+2b) \\ x &= X \\ y &= Y \end{aligned} \tag{7.8}$$

sends the line-element into the form

$$ds^2 = -T(T+2b)(dX^2+dY^2) - (T+b)^2\,dZ^2 + dT^2 \tag{7.9}$$

with

$$\begin{aligned} u_a &= \delta_a^T, \qquad u^a = \delta_T^a \\ p &= -T^{-1}(T+2b)^{-1} + b^2T^{-2}(T+2b)^{-2} \\ A - p &= 4T^{-1}(T+2b)^{-1} \end{aligned} \tag{7.10}$$

The restriction $r > 0$ implies $T > 0$ or $T < -2b$. We consider the first possibility.

This line-element describes a *homogeneous* anisotropic universe of the type studied by Doroshkevich (1965), Kantowski & Sachs (1966) and Thorne (1967). In this particular model the universe emerges from a barrel type singularity at $T = 0$ [$T(T+2b) \to 0$, $(T+b)^2 \to b^2$ as $T \to 0^+$, see Thorne, 1967; Jacobs, 1968; MacCallum, 1971]. The fluid has zero acceleration, and the shear and expansion scalars [see (7.7)] are given by

$$\sigma_{ab}\sigma^{ab} = \tfrac{1}{6}b^2[T(T+b)(T+2b)]^{-2}$$

$$\theta^2 = (T+b)^{-2}[3 + 2b^2T^{-1}(T+2b)^{-1}]^2$$

As $T \to 0^+$, $\sigma_{ab}\sigma^{ab}$ and θ^2 both behave as T^{-2}, while as $T \to \infty$, $\sigma_{ab}\sigma^{ab}$ behaves as T^{-6} and θ^2 as T^{-2}. This confirms the fact (suggested by the form (7.9) of the line-element) that the expansion of this universe is initially highly anisotropic but as $T \to \infty$ it becomes isotropic $[\sigma_{ab}\sigma^{ab}/\theta^2 \to 0$ as $T \to \infty]$. One thus regards (MacCallum, 1971) the universe as becoming asymptotically (i.e. as $T \to \infty$) Friedmann (Robertson-Walker). The form of this line-element near the singularity ($T = 0$) has been given by Doroshkevich (1965) [see p. 141, equation (14) with $\lambda = \frac{1}{2}$] and by Thorne (1967) [see p. 64, equation (A7a) with $\gamma = 1$].

The pressure and energy density satisfy an equation of state of the form

$$4(A + 3p) - b^2(A - p)^2 = 0$$

As $T \to 0^+$ this approaches the form $p = A$, and as $T \to \infty$, the form $p = -\frac{1}{3}A$, which entails the unsatisfactory feature of negative pressure [see (7.10)].

In view of the discussion of this section, it seems reasonable to interpret the line-element (7.4) with $b^2 \neq$ constant (and the more general line-elements of Section 6) as *inhomogeneous*, anisotropic model universes. It is hoped that these solutions will provide examples in which one can study the singularities and time evolution of such model universes.

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